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Competition between glassiness and order in a multispin glass

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A mean-field multispin interaction spin glass model is analyzed in the presence of a ferromagnetic coupling. The static and dynamical phase diagrams contain four phases (paramagnet, spin glass, ordinary ferromagnet, and glassy ferromagnet) and exhibit reentrant behavior. The glassy ferromagnet phase has anomalous dynamical properties. The results are consistent with a nonequilibrium thermodynamics that has been proposed for glasses. [S1063-651X(99)51108-2]

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Recent work has emphasized the importance of aging as a fundamental property of glassy systems and given it a precise meaning: While simple average quantities relax relatively quickly to stationary values, two-time quantities (correlation and response functions) show that the system never truly equilibrates. Whenever the separation between the two times is comparable with the age of the system or greater, time translational invariance is violated and the fluctuation-dissipation relation (FDR) is modified. These effects are a direct consequence of trapping in metastable attractors, and the modified FDR gives information about the overlap distribution of these attractors. This has been demonstrated explicitly for several soluble models [1–3].

However, in all these models the distribution of the quenched random variables is symmetric, so the metastable attractors lack interesting macroscopic structure. This is not the case in a large variety of systems where biased, tuned, or trained interactions lead to cooperatively ordered attractors, yet this macroscopic ordering competes with significant quenched randomness and the consequent tendency toward glassiness. Examples of such systems include models for recurrent neural networks [4], error correction algorithms [5], combinatorial optimization [6–8], and proteins [9], as well as experimental spin glass materials with ferromagnetic, antiferromagnetic, or helically ordered phases [10]. Equilibrium analyses of many of these have found replica symmetry breaking regions, indicative of glassy behavior, in their phase diagrams, but so far their dynamics have not been studied.

The purpose of this Rapid Communications is to start to remedy this lack by solving a nontrivial model with both spin-glass-like and macroscopic attractors. We demonstrate that (i) the ferromagnetic part of the dynamical phase diagram contains both glassy and ordinary nonglassy regions (with aging present in the glassy one), (ii) the two-time correlation function in the glassy ferromagnet involves nonanalytic features absent in the zero-field spin glass [3]. The results are in accord with a proposed nonequilibrium thermodynamic description of glasses [11–13].

For the model we discuss, a spherical p-spin glass [1,14–16] with a ferromagnetic interaction, we observe several interesting features of the phase diagrams (Fig. 1). (i) For both statics and dynamics, they contain four phases: paramagnet, spin glass (with zero spontaneous magnetization), conventional ferromagnet (nonzero spontaneous magnetization but no replica symmetry breaking or aging), and glassy ferromagnet (nonzero spontaneous magnetization with replica symmetry breaking for statics and aging for dynamics). All the phases meet at a multicritical point. (ii) The critical ferromagnetic exchange separating the spin-glass and glassy ferromagnetic regions decreases with increasing temperatures, so that within a finite band of exchange interaction values there occurs a sequence of phases, with decreasing temperature, of paramagnet, ordinary ferromagnet, glassy ferromagnet and spin glass. This has been a regularly observed feature of experiment (referred to as ''re-entrance'' [10]), but is not found in equilibrium theory for conventional spin-glass models [17]. (iii) There is a finite maximum ferromagnetic exchange for glassy ferromagnetism even at zero temperature for all finite $p>2$ [18–20]. (iv) The transition temperature separating glassy and ordinary ferromagnetic regions first rises and then falls as the ferromagnetic exchange is increased beyond its value at the multicritical point, thereby indicating re-entrance as the ferromagnetic exchange is increased within an appropriate temperature band, with the sequence paramagnet, ordinary ferromagnet, glassy ferromagnet, and back to ordinary ferromagnet. (v) The peak of the phase line separating glassy and ordinary ferromagnet also marks a boundary between two types of onset of one-step replica symmetry breaking (1RSB), discontinuous for smaller ferromagnetic exchange, continuous for larger ferro-

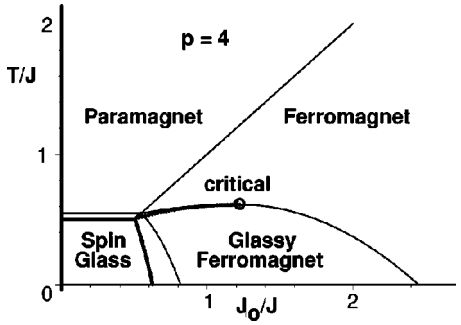

FIG. 1. Static and dynamic phase diagram for the model with $p=4$. When different from the dynamical ones, the static phase boundaries are indicated by bold lines. J_0 is the ferromagnetic coupling, and J is the variance of the spin-glass couplings.

magnetic exchange; cf. [14]. (vi) Wherever the onset of 1RSB is discontinuous, the dynamical transition temperature is higher than the static one, as in the limit of zero ferromagnetic interaction [14,15,19].

We consider a model Hamiltonian

$$\mathcal{H}=-\sum_{i_1<i_2..<i_p} J_{i_1 i_2..i_p} S_{i_1} S_{i_2}..S_{i_p} - \frac{J_0}{N}\sum_{ij} S_i S_j - H\sum_i S_i, \tag{1}$$

with independently distributed random quenched p-spin interactions of mean zero and variance $J^2 p!/2N^{p-1}$ and nonrandom two-spin interactions. The spins are subject to the spherical constraint $\Sigma_i S_i^2 = N$. Mean field theory is exact for infinite-ranged interactions. The choice of spherical spins simplifies the resulting self-consistency equations, while $p > 2$ ensures that one-step replica-symmetry breaking (1RSB) is sufficient.

We have studied the model by two complementary approaches. The first employs the replica formalism and permits us to obtain both the equilibrium and dynamical order parameters. It is characterized by three order parameters; the maximum (self-) overlap q_1, the minimum (mutual) overlap q_0, the magnetization M, and the amplitude $(1-x)$ of the self-overlap part of the overlap probability distribution. The spherical constraint is ensured by a self-consistently determined Lagrange multiplier. Stationarity of the replica free energy

$$\begin{aligned} F = &-\tfrac{1}{2}J_0 M^2 - \tfrac{1}{4}\beta J^2[1-(1-x)q_1^p - x q_0^p] \\ &- HM - \tfrac{1}{2}(T/x)\log[1-(1-x)q_1 - x q_0] \\ &+ \frac{(1-x)T}{2x}\log(1-q_1) + \frac{(M^2-q_0)T}{2(1-(1-x)q_1 - x q_0)} \end{aligned} \tag{2}$$

with respect to q_0, q_1, and M yields the self-consistency equations

$$M = (\beta H + \beta J_0 M)(1-\overline{q}), \tag{3}$$

$$q_0 = \mu(1-\overline{q})^2 q_0^{p-1} + M^2, \tag{4}$$

$$q_1 - q_0 = \mu(1-\overline{q})(1-q_1)(q_1^{p-1} - q_0^{p-1}), \tag{5}$$

where we have used the shorthands $\mu = \frac{1}{2}p\beta^2 J^2$ and $\overline{q} = x q_0 + (1-x)q_1$.

For the equilibrium (static) theory, a fourth self-consistency condition is provided by requiring that the derivative

$$\begin{aligned} \frac{\partial F}{\partial x} = \frac{T}{2}\Bigg[&\frac{1}{x^2}\log\frac{1-\overline{q}}{1-q_1} - \frac{q_1-q_0}{x(1-\overline{q})} - \frac{\beta^2 J^2}{2}(q_1^p - q_0^p) \\ &- \frac{(M^2-q_0)(q_1-q_0)}{(1-\overline{q})^2}\Bigg] \end{aligned} \tag{6}$$

vanish. Equations (3)–(6) are then solved for q_0, q_1, M, and x.

To obtain the dynamical order parameters one employs, instead of Eq. (6), the marginal stability condition [14,21,22]

$$(p-1)\mu q_1^{p-2}(1-q_1)^2 = 1. \tag{7}$$

As in the problem without a ferromagnetic term [14], this procedure yields the same order parameters and transitions that we find with our second approach, a direct dynamical analysis.

That treatment starts from the Langevin equation

$$\frac{\partial S_i}{\partial t} = -\frac{\partial \mathcal{H}}{\partial S_i} - z(t)S_i + \eta_i(t), \tag{8}$$

where $\eta_i(t)$ is white noise of temperature T and $z(t)$ has to be adjusted to satisfy the spherical constraint. Following and extending now-standard procedures [23] of introducing a generating functional, averaging over stochastic noise and quenched disorder, introducing appropriate macroscopic time-dependent quantities and using extremal analysis in the limit $N\to\infty$, there result self-consistent equations for the local correlation function $C(t,t') = (1/N)\Sigma_i \langle S_i(t) S_i(t')\rangle$, the local response function $G(t,t') = (1/N)\Sigma_i \delta\langle S_i(t)\rangle / \delta H_i(t')|_{H_i(t')=H}$, and the global magnetization $M(t) = (1/N)\Sigma_i \langle S_i(t)\rangle$:

$$\begin{aligned} \partial_t C(t,t') = &-z(t)C(t,t') + 2G(t',t) + \beta H M(t') \\ &+ \beta J_0 M(t) M(t') + \mu \int_0^{t'} dt_1 C^{p-1}(t,t_1) G(t',t_1) \\ &+ (p-1)\mu \int_0^t dt_1 G(t,t_1) C^{p-2}(t,t_1) C(t_1,t'), \end{aligned} \tag{9}$$

$$\partial_t G(t,t') = -z(t)G(t,t') + \delta(t-t') + (p-1)\mu \times \int_{t'}^t dt_1 G(t,t_1) C^{p-2}(t,t_1) G(t_1,t'), \tag{10}$$

$$\partial_t M(t) = -z(t)M(t) + \beta H + \beta J_0 M(t) + (p-1)\mu \times \int_0^t dt_1 G(t,t_1) C^{p-2}(t,t_1) M(t_1). \tag{11}$$

Together with the spherical constraint $C(t,t) = 1$, Eqs. (9)–(11) determine the dynamics completely. However, even for $J_0 = H = 0$, they have not yet been solved. We concentrate on long times, where $z(t)$ and $M(t)$ reach stationary values and a self-consistent solution is possible under the assumption that $C(t,t')$ and $G(t,t')$ have time-translation-invariant behavior for $(t-t') \ll t'$ and simple aging behavior for $(t-t') \gg t'$ [1]:

$$C(t,t') = C_{st}(t-t') + C_{ag}(t,t'), \tag{12}$$

$$G(t,t') = G_{st}(t-t') + G_{ag}(t,t'), \tag{13}$$

where in terms of $\lambda = t'/t$, $C_{ag}(t,t') = \mathcal{C}(\lambda)$ and $G_{ag}(t,t') = \mathcal{G}(\lambda)/t$, with limiting values $C_{st}(0) = 1-q_1$, $C_{st}(\infty) = 0$, $\mathcal{C}(1) = q_1$, and $\mathcal{C}(0) = q_0$. Thus, q_1 is the plateau value of C reached for $1 \ll t-t' \ll t'$, and q_0 is its asymptotic $(t\to\infty)$ limit. In the stationary regime the conventional fluctuation-dissipation theorem

$$\frac{\partial C_{st}(t-t')}{\partial t'} = TG_{st}(t-t') \quad (t>t') \tag{14}$$

holds, while in the aging regime one has instead the modified fluctuation-dissipation relation

$$\frac{\partial C_{ag}(t,t')}{\partial t'} = T_e G_{ag}(t,t'), \quad T_e = \frac{T}{x}, \tag{15}$$

i.e., $d\mathcal{C}(\lambda)/d\lambda = T_e \mathcal{G}(\lambda)$.

At long times the time-derivative terms on the left-hand sides of Eqs. (9)–(11) can be neglected. The asymptotic solution found in this limit for the aging regime admits a reparametrization invariance [24]: if $C(t,t')$ is a solution, so is $C[h(t),h(t')]$, with $h(t)$ an arbitrary monotonic function of t. Thus we cannot find the complete time-dependence of C and G from the asymptotic equations alone. Nevertheless, we can solve for M, x, q_0, and q_1, finding the same results as were obtained above by the replica treatment in its ''dynamics'' form. No further assumptions on C and G are needed to obtain these results.

We restrict ourselves to $H=0$ in this Rapid Communication. Figure 1 shows the phase diagram of the model in $T-J_0$ space for $p=4$. The general features are not sensitive to p. For sufficiently weak ferromagnetic interaction J_0, we find the same results as for $J_0=0$: a dynamical paramagnetic-to-spin glass transition at a temperature T_d and a static transition at a lower temperature T_g. The spin glass states (both dynamical and equilibrium) involve 1RSB, with $q_0=M=0$, and q_1 is discontinuous at the transitions, where $x\to 1$. For $J>T_d$ (T_g for statics), there is a Curie temperature T_c, below which the paramagnetic state is unstable against the onset of spontaneous magnetization. For a range of temperatures below T_c this ferromagnetic state has a nonzero spin glass order parameter, but no glassy properties (no RSB or aging): $q_1=q_0$. However, it is unstable, at low enough temperatures and not too large J_0, against the formation of a glassy ferromagnetic state with nonzero M, q_0, and $q_1>q_0$ (which implies aging). Below a temperature-dependent critical value of J_0, the ferromagnetism of this state becomes unstable, and one recovers the simple spin glass phase. All four phases come together at the point $J_0=T=T_d$ ($J_0=T=T_g$ for statics).

The upper boundary of the glassy ferromagnetic phase rises as J_0 increases from T_d (T_g for statics) and reaches a maximum at $J_0/J=\sqrt{p(p-1)/2}[(p-2)/p]^{(p-2)/2}$. It falls to $T=0$ at $J_0/J=\sqrt{p(p-1)/2}$. To the left of the maximum, q_1-q_0 jumps discontinuously and $x\to 1$ at the transition (as in the paramagnetic-to-spin glass transition at small J_0). To the right of the maximum, q_1-q_0 goes continuously to zero at the transition, where $x<1$, and the static and dynamical boundaries coincide. This part of the boundary is thus an Almeida-Thouless line like that found in the SK model [25], though the low-T states are different: here one step of RSB is exact, while full RSB is necessary in the SK model. The overall shape of the boundary is similar to that in the p-spin glass in an external field [14,15], though the relation between the two phase diagrams is not elementary because the order parameter equations are coupled.

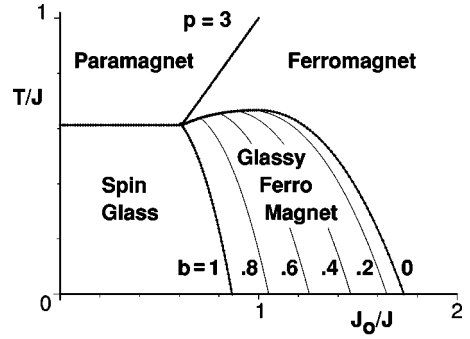

FIG. 2. Dynamical phase transition lines (solid) for the model with $p=3$, and lines of constant b in the glassy ferromagnet.

In the large-p limit, the dynamical small-J_0 spin glass transition temperature T_d approaches a finite limit $1/\sqrt{2e}$. The glassy ferromagnetic phase extends out to $J_0=Jp/\sqrt{2}$, with its maximum temperature, $2T_d/\sqrt{e}$ achieved at $J_0=Jp/(e\sqrt{2})$. The *static* spin glass transition temperature T_g at small J_0 goes to zero like $1/\sqrt{\log p}$ [16], so the spin glass phase disappears, but the glassy ferromagnetic phase remains.

The glassy state is characterized by a plateau $C=q_1$ in the correlation function, the length of which is age-dependent. The behavior near this plateau involves power laws. An exponent a characterizes the decay at the end of the short-time regime (the approach to the plateau): $C(t+\tau,t)\equiv C(\tau)\approx q_1+\text{const}/\tau^a$. Using Eq. (10) in the FDT regime, we find an equation for a,

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{(p-2)(1-q_1)}{2q_1}, \tag{16}$$

valid in both glassy ferromagnetic and spin glass phases. Thus, a is independent of J_0, since q_1 is fixed by Eq. (7).

For $J_0=0$ the asymptotic dynamical equations have the exact aging solution $\mathcal{C}(\lambda)=\lambda^\nu$ $(0<\nu<1)$ [1], which also holds for $J_0>0$ in the spin glass phase. In the glassy ferromagnetic phase, $\mathcal{C}(\lambda)$ is nonanalytic for $\lambda\to 1$, i.e., at the end of the plateau and the beginning of the aging regime. We make the Ansatz $\mathcal{C}(\lambda)=1-B(1-\lambda)^b+\mathcal{O}[(1-\lambda)^{2b}]$ and find, using the result (16), that the exponent b must satisfy

$$x\frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \frac{\Gamma^2(1-a)}{\Gamma(1-2a)}. \tag{17}$$

Such an Ansatz was also employed in a different problem [3], where the relation (17) was also found. A similar result is expected to hold for a spin glass in a field, as well. At the boundary between the spin glass and glassy ferromagnetic phases, $b\to 1$, and along the AT line separating the conventional and glassy ferromagnetic phases $b\to 0$. Figure 2 shows lines of constant b for the $p=3$ model.

The system dynamically condenses into glassy states, characterized by x or the effective temperature $T_e=T/x$. They have a configurational entropy (or complexity) [11,12,26,27] $\mathcal{I}=-(\partial F/\partial T_e)_T=(x^2/T)(\partial F/\partial x)_T$, that follows from Eq. (6). $\mathcal{I}$ is a positive constant in the spin glass [26], and, of course, it is zero in the ordinary ferromagnet. In the glassy ferromagnet it interpolates smoothly between

these two values, and it vanishes at the transition to the ordinary ferromagnet, where $q_0 \to q_1$.

The specific heat, defined as the limit of the energy difference between states obtained by rapid quenches to two slightly different temperatures, divided by the temperature difference, has two terms: $C = dU/dT = T dS_1/dT + T_e d\mathcal{I}/dT$ [11,12], with the intravalley entropy given by $S_1 = -(\partial F/\partial T)_{T_e}$, viz.,

$$S_1 = \tfrac{1}{2}\ln(1-q_1) - \tfrac{1}{4}(\beta J)^2[1+(p-1)q_1^p - p q_1^{p-1}]. \tag{18}$$

This is exactly the entropy of a single TAP valley [26,28] and describes states that can be reached dynamically at fixed T. In the spin glass and in the ordinary ferromagnet $d\mathcal{I}/dT = 0$, so the specific heat merely follows from the intravalley processes. In the glassy ferromagnet, however, $d\mathcal{I}/dT < 0$, so the specific heat (as defined here) acquires a (negative) contribution from changes in the shape of the free energy landscape with temperature.

In summary, we have been able to elucidate explicitly the consequences of the competition between glassiness and ferromagnetic ordering in the statistical mechanics and long-time dynamics of an asymptotically soluble model. We have found several features and extended and verified the applicability of concepts devised for spin glasses. These results can shed useful light on the many other important problems, in physics and other fields, where ordering competes with quenched disorder.

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